On free-surface oscillations in a rotating paraboloid

By JOHN W. MILES
Department of Mathematics, Institute of Advanced Studies,
Australian National University, Canberra

AND F. K. BALL
C.S.I.R.O. Division of Meteorological Physics,
Aspendale, Victoria, Australia

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Lamb's analysis of small-amplitude, shallow-water oscillations in a rotating paraboloid, interpreted by him in the inconsistent context of an approximately plane free surface, is re-interpreted to obtain results that are valid for

$$0 \leqslant \omega^2 l/2g < 1$$

(ω = rotational speed, l = latus rectum of paraboloid); no equilibrium is possible for $\omega^2 l/2g > 1$. It is shown that the frequencies of the dominant modes for the azimuthal wave numbers 0 (axisymmetric motion) and 1 are independent of ω for an observer in a non-rotating reference frame and that the frequencies of all other axisymmetric modes are decreased by rotation (Lamb concluded that they would be increased). An axisymmetric mode of zero frequency, which was overlooked by Lamb, also is found.

Exact solutions to the non-linear equations of motion, which reduce to the aforementioned dominant modes for small amplitudes, are determined. The axisymmetric solution is inferred from similarity considerations and is found to contain all harmonics of the fundamental frequency. The finite motion of azimuthal wave-number 1 is a quasi-rigid displacement of the liquid and is found to be simple harmonic except for a second-harmonic component of the free-surface displacement (but the horizontal velocity at a given point remains simple harmonic).

1. Introduction

Lamb (1932, §§210 and 212), considering small-amplitude, shallow-water oscillations in both a rotating circular cylinder and a rotating paraboloid and proceeding from the hypothesis that the free surface could be approximated by a plane (in which approximation the angular velocity ω enters the calculation only through the Coriolis acceleration), concluded that the angular frequencies (σ) of axisymmetric oscillations are given by

$$\sigma^2 = \sigma_0^2 + 4\omega^2, \tag{1.1}$$

where σ_0 denotes the angular frequency of a given mode in the absence of rotation. We add that the hypothesis of a plane free surface leads to (1.1) for axisymmetric oscillations in any surface of revolution. We also note that Lamb's (§ 210) analysis is correct for 'uniform depth' (paraboloidal bottom congruent with free surface).

Fultz (1962) has recently observed that (1.1) is incorrect in consequence of the implicit neglect of the free-surface slope in its derivation. He was led to this conclusion in the first instance by a series of experiments with flat-bottomed circular cylinders, which yielded positive values of $(\sigma^2-\sigma_0^2)/4\omega^2$ that were appreciably less than one and decreased monotonically with each of d/a and ω^2a^2/gd (a = radius and d = mean depth). Subsequently, Platzman (1962) calculated the limiting value (for the dominant mode)

$$\frac{\sigma^2-\sigma_0^2}{4\omega^2} \to 0{\cdot}69 \quad \left(\frac{d}{a} \to 0, \frac{\omega^2a^2}{gd} \to 0\right). \tag{1.2}$$

Miles (1963) has calculated $(\sigma^2-\sigma_0^2)/4\omega^2$ *vs* d/a for $\omega^2a^2/gd \to 0$ and obtained numerical results in agreement with those measured by Fultz.

The solution of the shallow-water equations in a circular cylinder rotating at a non-small speed, between zero and that value at which the paraboloidal free surface touches bottom, can be expressed in terms of hypergeometric functions (cf. Lamb, § 212), but the determination of the natural frequencies requires the solution of two, simultaneous, transcendental equations. The corresponding problem for a paraboloidal container is much simpler, in that the solution can be expressed in terms of hypergeometric polynomials with an algebraic frequency equation. We shall show here that Lamb's solution to the latter problem, interpreted by him in the context of an approximately plane free surface, can be appropriately re-interpreted for the entire range of admissible ω by virtue of the fact that the radial variation of depth remains parabolic.

We shall find not only that (1.1) is incorrect for a paraboloid (a trivial extension of Fultz's remark), but also that the frequencies of the dominant modes with azimuthal wave-numbers 0 (axisymmetric mode) and 1 are independent of ω for an observer in a non-rotating reference frame. We also shall show that the frequencies of all other axisymmetric modes are *decreased* by rotation, so that (1.1) is wrong qualitatively, as well as quantitatively.

[The axisymmetric, gravity-wave modes in a rotating, circular cylinder with a paraboloidal bottom have recently been determined by Murty (1962). His results include the flat-bottomed cylinder and the paraboloid as special cases; in particular, he obtained a result equivalent to (3.6) below. He also carried out experiments and confirmed the theoretical prediction that the frequency of the dominant, axisymmetric mode in a rotating paraboloid is independent of the rotational speed. We are indebted to Prof. Fultz for bringing Murty's thesis to our attention.]

That the frequencies of two particular modes are independent of ω in a non-rotating reference frame suggests that each must have a special simplicity. In fact, they are special cases of two general classes of solutions to the non-linear, shallow-water equations (Ball 1962, 1963). As we shall show, the axisymmetric mode of finite amplitude may be inferred from a similarity solution to the non-linear equations and contains all harmonics of the fundamental frequency. The finite-amplitude mode of azimuthal wave-number 1 is even simpler in that the liquid exhibits a quasi-rigid motion that is simple harmonic except for a second harmonic in the free-surface displacement (but the horizontal velocity at a particular point remains simple harmonic).

2. Formulation

We consider a paraboloid of latus rectum l that contains a volume of liquid $\frac{1}{2}\pi l d^2$ and rotates (together with its contents) with angular velocity ω about a vertical axis of symmetry. The liquid would have a plane free surface and a maximum depth d under the action of gravity alone, but the equilibrium free surface under the joint action of gravity and centrifugal force† has the paraboloidal shape

$$z = h_0 + \tfrac{1}{2}(\omega^2/g)\, r^2, \tag{2.1}$$

where z and r are cylindrical polar co-ordinates with origin at the vertex of the container, and h_0 is the maximum depth. Invoking the constraint of constant volume, we obtain

$$h_0 = (1-\alpha)^{\frac{1}{2}}\, d, \tag{2.2}$$

where

$$\alpha = \omega^2 l/2g \quad (0 \leqslant \alpha < 1). \tag{2.3}$$

The radial variation of depth then is given by

$$h(r) = h_0 + \tfrac{1}{2}(\omega^2/g)\, r^2 - l^{-1} r^2 \tag{2.4a}$$

$$= h_0[1-(r/a)^2], \tag{2.4b}$$

where

$$a = (1-\alpha)^{-\frac{1}{4}}\,(ld)^{\frac{1}{2}} \tag{2.5}$$

is the value of r at the intersection of the free surface with the underlying surface of the container. These two surfaces coalesce, and are infinite in extent, for $\alpha = 1$; no equilibrium is possible for $\alpha > 1$.

We note that the lowest frequency of shallow-water oscillations in the absence of rotation is given by (Lamb, § 193)

$$\sigma_1^2 = 2g/l, \tag{2.6}$$

so that, from (2.3),

$$\alpha = (\omega/\sigma_1)^2. \tag{2.7}$$

The linearized, shallow-water approximations to the continuity and Euler equations are (Lamb, § 209)

$$r\zeta_t + (rhu)_r + (hv)_\theta = 0, \tag{2.8}$$

$$u_t - 2\omega v + g\zeta_r = 0, \tag{2.9}$$

$$v_t + 2\omega u + g r^{-1}\zeta_\theta = 0, \tag{2.10}$$

where ζ denotes the free-surface displacement, u and v denote the radial and tangential components of the velocity, r and θ denote polar co-ordinates in a

† The effects of Coriolis acceleration and free-surface curvature in the analysis of §§ 2 and 3 could be separated by replacing ω^2/g on the right-hand side of (2.1) by the curvature $1/R$. The results would correspond to an equilibrium free surface in a non-uniform gravitational field; $R < 0$ would imply a convex free surface, as in large-scale oceanographic applications. The analysis of §§ 2 and 3, for which $\omega^2 R/g = 1$, refers to normal laboratory configurations.

reference frame that rotates with the container, and subscripts imply partial differentiation. The tangential velocity and polar angle in the non-rotating reference frame are $v+\omega r$ and $\theta+\omega t$.

Lamb's solution of (2.4*b*) and (2.8)–(2.10) yields the eigenfunctions

$$\zeta_{sj} = A_{sj}(r/a)^s F[s+j, 1-j;\, s+1;\, (r/a)^2]e^{i(\sigma t+s\theta)} \qquad (s = 0,1,2,\ldots;\, j = 1,2,\ldots), \quad (2.11)$$

where F is a hypergeometric polynomial and the angular frequency σ is determined by

$$\frac{(\sigma^2-4\omega^2)\,a^2}{gh_0} - \frac{4\omega s}{\sigma} = 2[(2j-1)\,s+2j(j-1)]. \quad (2.12)$$

The corresponding velocity is given by

$$\mathbf{q} = \{u, v\} = \left(\frac{i\sigma g}{\sigma^2-4\omega^2}\right)\left[\left\{\zeta_r, \frac{is\zeta}{r}\right\} + \left(\frac{2\omega}{\sigma}\right)\left\{\frac{s\zeta}{r}, i\zeta_r\right\}\right]. \quad (2.13)$$

We observe that both the relative depth, h/h_0, and the mode shape, qua functions of r/a, are independent of ω.

We emphasize that the effects of free-surface slope are implicitly incorporated in the solution of (2.11)–(2.13) by way of the equation of continuity. However, Lamb's subsequent deductions from (2.12) were made in the context of an approximately plane free surface and of fixed (independently of ω) h_0 and a and therefore are quite misleading. We also remark that the shallow-water approximation demands

$$h_0/l = (1-\alpha)^{\frac{1}{2}}\,(d/l) \ll 1, \quad (2.14)$$

but this does not necessarily imply $d \ll l$.

3. The frequency equation

Substituting h_0 and a from (2.2) and (2.5) into (2.12), we may rewrite the result in the form

$$f(\sigma,\omega) = \sigma^3 - [4\omega^2 + (1-\alpha)\,\sigma_{sj}^2]\,\sigma - 2(1-\alpha)\omega\sigma_s^2 = 0, \quad (3.1)$$

where

$$\sigma_s^2 \equiv \sigma_{s1}^2 = s\sigma_1^2 = 2sg/l \quad (3.2)$$

and

$$\sigma_{sj}^2 = [(2j-1)\,s+2j(j-1)]\,\sigma_1^2. \quad (3.3)$$

Following Lamb (§ 223), we designate those oscillations for which $|\sigma| > 2\omega$ as modes of the first class and those for which $|\sigma| < 2\omega$ as modes of the second class. We may assume $\omega > 0$ without loss of generality; if $\omega < 0$ we have only to replace σ by $-\sigma$ in (3.1).

We consider first axisymmetric oscillations, for which (3.1) reduces to

$$f(\sigma,\omega) = \sigma[\sigma^2 - 4\omega^2 - (1-\alpha)\sigma_{0j}^2] \quad (s = 0). \quad (3.4)$$

The root $\sigma = 0$ is trivial if $j \neq 2$, but if $j = 2$ it leads to a non-trivial solution of (2.8)–(2.10), which we may write in the normalized form

$$u = 0, \quad v = \delta\omega r, \quad \zeta = -\left(\frac{\alpha}{1-\alpha}\right)\delta h_0\left[1-2\left(\frac{r}{a}\right)^2\right] \quad (s = 0, j = 2, \sigma = 0). \quad (3.5)$$

We may identify this mode of the second class as a perturbation of the equilibrium motion associated with a perturbation $\delta\omega$ in the angular velocity. It is an example of that class of steady motions to which Lamb refers at the end of § 207; however, his subsequent† derivation of the partial-differential equation § 212 (2) and the eigenvalue equation § 212 (4) involves division by σ and thereby precludes the solution (3.5).

The non-zero roots of (3.4) are given by

$$\begin{aligned}\sigma^2 &= (1-\alpha)\,\sigma_{0j}^2+4\omega^2\\ &= \sigma_{0j}^2-2(j+1)(j-2)\,\omega^2 \quad (s=0;\, j=2,3,\ldots)\end{aligned} \tag{3.6}$$

and clearly yield modes of the first class. The solution implied by $s=0, j=1$ is trivial: $\zeta_{01}\equiv 0$; $\zeta_{01}=$ const. would violate the constraint of constant volume.

We conclude from (3.6) that the frequency of the dominant axisymmetric mode ($j=2$) is unaffected by rotation. If $j\geqslant 3$, σ^2 decreases linearly with ω^2 from σ_{0j}^2 at $\alpha=0$ to $(2\omega)^2$ at $\alpha=1$. Lamb's conclusion that σ^2 increases with ω^2 follows from the implicit neglect of α compared with 1 in (3.4), in which approximation (3.4) yields (1.1) above.

We consider next that class of oscillations for which $j=1$, which reduces (2.11) and (3.1) to

$$\zeta_{s1} = A_{s1}(r/a)^s\, e^{i(\sigma t+s\theta)}, \tag{3.7}$$

$$f(\sigma,\omega) = (\sigma+2\omega)\,[\sigma^2-2\omega\sigma-(1-\alpha)\,\sigma_s^2] = 0. \tag{3.8}$$

The root $\sigma=-2\omega$ renders (2.13) indeterminate, but otherwise (3.7) implies

$$\{u,v\} = ig(\sigma-2\omega)^{-1}\{1,i\}\,(s\zeta/r). \tag{3.9}$$

Substituting (3.7) and (3.9) into the equation of continuity, (2.8), we find that they constitute a non-trivial solution for $\sigma=-2\omega$ if and only if

$$\alpha=\alpha_1=s/(s+8), \tag{3.10}$$

in which case $\sigma=-2\omega$ is a double root of (3.8). If $\alpha\neq\alpha_1$, $\sigma=-2\omega$ yields only the trivial solution $\zeta,u,v\equiv 0$. (Lamb discarded the factor $\sigma+2\omega$ in (3.8) as extraneous, which it is, but he did not mention the existence of a non-trivial solution for $\sigma=-2\omega$. His formulation yields such a non-trivial solution for $\alpha=\frac{1}{8}s$, rather than $s/(s+8)$.)

The remaining roots of (3.8) are given by

$$\begin{aligned}\sigma &= \omega\pm[(1-\alpha)\,\sigma_s^2+\omega^2]^{\frac{1}{2}}\\ &= \omega\pm[\sigma_s^2-(s-1)\,\omega^2]^{\frac{1}{2}} \quad (s=1,2,\ldots;\, j=1).\end{aligned} \tag{3.11}$$

These roots descend to the gravity-wave frequencies $\pm\sigma_s$ at $\alpha=0$ and to $2\omega+$ and $0-$ at $\alpha=1-$. The larger frequency lies in the interval $(2\omega,\infty)$ for all α in

† Lamb's statement that 'the contour-lines of the free surface must be everywhere parallel to the contour-lines of the bottom' if such steady motions are to exist is somewhat misleading, if not inaccurate. It follows directly from the equations of motion (2.8)–(2.10) that a steady motion can be derived from a stream function $g\zeta/2\omega$ provided that: (*a*) $\nabla\zeta$ is chosen proportional to ∇h and (*b*) the boundary conditions can be satisfied. Lamb's restriction, as quoted above, evidently is sufficient but not necessary for the satisfaction of (*a*).

(0, 1) and therefore always corresponds to a mode of the first class. The smaller frequency lies in $(-\sigma_s, -2\omega)$ for α in $(0, \alpha_1)$ and in $(-2\omega, 0)$ for α in $(\alpha_1, 1)$, corresponding respectively to modes of the first and second class.

The frequencies given by (3.11) are for an observer in the rotating reference frame. The corresponding frequencies for an observer in the non-rotating frame are

$$\sigma - s\omega = -(s-1)\,\omega \pm [\sigma_s^2 - (s-1)\,\omega^2]^{\frac{1}{2}}; \tag{3.12}$$

in particular,

$$\sigma - \omega = \pm\sigma_1 \quad (s = j = 1). \tag{3.13}$$

We conclude that the frequency of the dominant mode for $s = 1$, as measured in a non-rotating reference frame, is independent of ω. We also remark that

$$\sigma - s\omega = 0,\ -2\omega \quad \text{at} \quad \alpha = (s-1)^{-1} \quad (s = 2, 3, \ldots; j = 1). \tag{3.14}$$

Finally, we consider the general case $s \geqslant 1$ and $j \geqslant 2$ in (3.1)–(3.3). Invoking Descartes' rule of signs and remarking that the coefficient of σ^2 is identically zero, we infer that the cubic equation (3.1) has three real roots of zero sum. Observing that

$$\operatorname{sgn} f = -1,\ +1,\ -1,\ -1,\ 1$$

for

$$\sigma = -\sigma_{sj},\ -2\omega,\ 0,\ 2\omega,\ \infty,$$

we infer that these roots lie in the intervals $(-\sigma_{sj}, -2\omega)$, $(-2\omega, 0)$, $(2\omega, \infty)$, corresponding to two modes of the first class and one of the second.

Expanding f about $\omega = 0$, where

$$f(\sigma, 0) = \sigma(\sigma^2 - \sigma_{sj}^2), \tag{3.15}$$

we find that the outer roots tend to the gravity-wave frequencies $\pm\sigma_{sj}$ and the inner root to zero according to

$$\sigma = \pm\sigma_{sj}[1 + O(\alpha)] + \gamma_{sj}\,\omega \tag{3.16a}$$

and

$$\sigma = -2\gamma_{sj}\,\omega[1 + O(\alpha)], \tag{3.16b}$$

where

$$\gamma_{sj} = s[s + 2(j+s)(j-1)]^{-1}. \tag{3.17}$$

Substituting (3.16*b*) into (2.13), we find that $\zeta = O(\omega\mathbf{q})$, from which we infer that a mode of the second class degenerates to an internal motion, unaccompanied by free-surface displacement, in the limit $\alpha \to 0$. Such motions, originally discovered by Kelvin (cf. Lamb, §§206, 212), are sometimes referred to as inertial modes.

Expanding f about $\alpha = 1$, where

$$f(\sigma, \sigma_1) = \sigma(\sigma^2 - 4\omega^2), \tag{3.18}$$

we find that the outer roots tend to $\pm 2\omega$ and the inner root to zero according to

$$\frac{\sigma}{2\omega} = \begin{Bmatrix} 1 + \frac{1}{4}j(s+j-1)(1-\alpha) \\ -1 - \frac{1}{4}(j-1)(s+j)(1-\alpha) \\ -\frac{1}{4}s(1-\alpha) \end{Bmatrix} + O(1-\alpha)^2. \tag{3.19}$$

Substituting (3.19) into (2.13), we find that $\zeta = O[(1-\alpha)\,\mathbf{q}]$ for the outer roots, and hence that the modes of the *first* class degenerate to inertial modes in the limit $\alpha \to 1$. We also remark that (2.14) is satisfied automatically in this limit, although not uniformly with respect to d/l.

Some numerical values of $\sigma/2\omega$ and $\sigma a(gh_0)^{-\frac{1}{2}}$ for $s = 1$ and $j = 1$ and 2 are given by Lamb (p. 327) for several values of the parameter

$$\beta = 4\omega^2 a^2/gh_0 = 8\alpha/(1-\alpha). \tag{3.20}$$

As we have shown—see (3.13) above—the results for $s = j = 1$ are independent of ω in a non-rotating reference frame. Renormalizing Lamb's results for $s = 1$, $j = 2$ and adding the limiting results for $\alpha = 0$ and 1, we obtain table 1.

α	$\sigma/2\omega$	σ/σ_{12}
0	$+\infty$	$+1$
	$-0{\cdot}143$	0
	$-\infty$	-1
$\frac{1}{5}$	2·889	0·977
	$-0{\cdot}125$	$-0{\cdot}042$
	$-2{\cdot}764$	$-0{\cdot}935$
$\frac{3}{7}$	1·874	0·927
	$-0{\cdot}100$	$-0{\cdot}050$
	$-1{\cdot}774$	$-0{\cdot}877$
$\frac{5}{6}$	1·183	0·814
	$-0{\cdot}040$	$-0{\cdot}028$
	$-1{\cdot}143$	$-0{\cdot}786$
1	1	0·755
	0	0
	-1	$-0{\cdot}755$

TABLE 1. The frequencies given by (3.1) for $s = 1, j = 2$.

4. Non-linear similarity solution

The striking simplicity of the foregoing results for the 02 and 11 modes naturally invites an investigation of the corresponding finite-amplitude motions. Let

$$h(r,\theta,t) = h_0[1-(r/a)^2]+\zeta(r,\theta,t) \tag{4.1}$$

be the instantaneous depth of the liquid and u and v the radial and tangential components of velocity in the rotating reference frame. The non-linear, shallow-water approximations to the continuity and Euler equations then are (as we may deduce from Lamb, § 207)

$$rh_t+(rhu)_r+(hv)_\theta = 0, \tag{4.2}$$

$$u_t+uu_r+(v/r)(u_\theta-v)-2\omega v = -g\zeta_r, \tag{4.3a}$$

$$v_t+uv_r+(v/r)(v_\theta+u)+2\omega u = -(g/r)\zeta_\theta. \tag{4.3b}$$

We consider first the finite-amplitude counterpart of the 02 mode. Setting $s = 0, j = 2, A_{02} = \epsilon h_0$, and $\sigma = 2\sigma_1$ (from (3.3)) in (2.11) and taking the real part of the result (the imaginary part yields a solution differing only in phase), we obtain

$$\zeta_{02} = \epsilon h_0[1-2(r/a)^2]\cos(2\sigma_1 t) \quad (|\epsilon| \ll 1). \tag{4.4}$$

Observing that the instantaneous free surface implied by substituting (4.4) into

(4.1) is also a paraboloid of revolution and that the corresponding velocities u and v are linear in r, we are led to try the similarity solution

$$h = \lambda(\tau)\, h_0[1 - \lambda(\tau)\,(r/a)^2] \quad (\lambda > 0), \tag{4.5}$$

$$\{u, v + \omega r\} = \sigma_1 r\{\mu(\tau), \nu(\tau)\}, \tag{4.6}$$

$$\tau = 2\sigma_1 t. \tag{4.7}$$

We remark that: (4.5) comprises (for $0 < \lambda < \infty$) all paraboloids consistent with the constraints of axial symmetry and of constant volume between the free surface and the underlying paraboloid $z = r^2/l$; $\lambda = 1$ implies the equilibrium free surface of (2.4); and (2.4), (4.1) and (4.5) yield

$$\zeta = h_0(\lambda - 1)\,[1 - (\lambda + 1)\,(r/a)^2]. \tag{4.8}$$

We also note that (4.6) gives the velocity in a non-rotating reference frame and that ω is arbitrary to the extent that the choice of an equilibrium free surface for a non-linear oscillation is arbitrary. Anticipating the existence of a periodic motion, we shall find it expedient to define ω as the (temporal) mean angular velocity. This implies, through (4.6), that

$$\bar{\nu} = \frac{1}{\sigma_1}\overline{\left(\frac{v}{r} + \omega\right)} \equiv \frac{\omega}{\sigma_1} = \alpha^{\frac{1}{2}}. \tag{4.9}$$

Substituting (4.5)–(4.8) into (4.2) and (4.3) and simplifying the resulting expression for $g\zeta_r$ with the aid of (2.2)–(2.7), we obtain

$$\lambda' + \mu\lambda = 0, \tag{4.10a}$$

$$2\mu' + \mu^2 - \nu^2 - (1 - \alpha)\,\lambda^2 + 1 = 0, \tag{4.10b}$$

$$\nu' + \mu\nu = 0. \tag{4.10c}$$

Regarding μ as a function λ, we may transform (4.10 a, b) to

$$\left(\frac{d}{d\lambda}\frac{\mu^2 + 1}{\lambda}\right) + 1 - \alpha + \left(\frac{\nu}{\lambda}\right)^2 = 0, \tag{4.11}$$

where $(\nu/\lambda)^2$ is constant by virtue of (4.10a, c).† Integrating (4.11) and (4.10a) by successive quadratures, we obtain

$$\lambda_0/\lambda = \nu_0/\nu = 1 + \tfrac{1}{2}[(1 - \alpha)\,\lambda_0^2 + \mu_0^2 + \nu_0^2 - 1]\,(1 - \cos\tau) + \mu_0 \sin\tau, \tag{4.12}$$

where λ_0, μ_0, ν_0 denote the initial values of λ, μ, ν. We may choose $\mu_0 = 0$ without loss of generality, measuring t from that instant at which $u = 0$. Taking the mean value of (4.12) and invoking (4.9), we obtain

$$\nu_0 = \alpha^{\frac{1}{2}}\lambda_0. \tag{4.13}$$

Finally, anticipating the subsequent reduction for small amplitudes, we introduce the parameter ϵ, such that

$$\lambda_0^2 = (1 + \epsilon)/(1 - \epsilon). \tag{4.14}$$

We then may reduce (4.12) to

$$\lambda = \alpha^{-\frac{1}{2}}\nu = (1 - \epsilon^2)^{\frac{1}{2}}\,(1 - \epsilon\cos\tau)^{-1} \tag{4.15}$$

† We note that ν/λ is the angular momentum of the motion divided by $\frac{1}{3}ma^2\sigma_1$ and that the first integral of (4.11) is the total energy of the motion divided by $\frac{1}{6}ma^2\sigma_1^2$, where m is the total mass of the liquid.

and integrate (4.10*a*) to obtain

$$\mu = \epsilon(1-\epsilon\cos\tau)^{-1}\sin\tau. \tag{4.16}$$

It is obvious that the non-linear oscillation described by (4.5)–(4.8), (4.15) and (4.16) is periodic with fundamental frequency $2\sigma_1$ and contains all harmonics of $2\sigma_1$. The mean value of the velocity in the rotating reference frame vanishes by virtue of our choice of ω, but the free-surface displacement has the non-zero mean value

$$\zeta = h_0[1-(1-\epsilon^2)^{-\frac{1}{2}}](r/a)^2, \tag{4.17}$$

in consequence of which the mean free surface lies below the equilibrium free surface for $r > 0$. We also note that the maximum and minimum radial displacements of the surface are given by

$$r_{max}/a = a/r_{min} = [(1+\epsilon)/(1-\epsilon)]^{\frac{1}{2}}. \tag{4.18}$$

The mean displacement given by (4.17) naturally has no significance for $r > r_{min}$.

To investigate the behaviour of the foregoing solution for small amplitudes, we may expand (4.15) and (4.16) in powers of ϵ and substitute the results into (4.6) and (4.8) to obtain

$$\lambda = 1+\epsilon\cos\tau+\tfrac{1}{2}\epsilon^2\cos 2\tau+\ldots, \tag{4.19}$$

$$\left\{\frac{u}{\sigma_1 r}, \frac{v}{\omega r}\right\} = \epsilon\{\sin,\cos\}\,\tau+\tfrac{1}{2}\epsilon^2\{\sin,\cos\}\,2\tau+\ldots, \tag{4.20}$$

$$\zeta = h_0\{-\tfrac{1}{2}\epsilon^2(r/a)^2+\epsilon[1-2(r/a)^2]\cos\tau+\tfrac{1}{2}\epsilon^2[1-3(r/a)^2]\cos 2\tau+\ldots\}. \tag{4.21}$$

The leading terms, of $O(\epsilon)$, in (4.20) and (4.21) evidently correspond to the 02 mode of (4.4).

5. Quasi-rigid solution

We turn now to the finite-amplitude counterpart of the 11 mode. Setting $s = j = 1$, $A_{11} = \epsilon_\pm h_0 \exp(\pm i\delta_\pm)$ (we now wish to permit arbitrary phases) and $\sigma = \omega \pm \sigma_1$ (from (3.11)) in (2.11) and taking the real part of the result, we obtain

$$\zeta_{11}^{(\pm)} = \epsilon_\pm h_0(r/a)\cos[(\theta+\omega t)\pm(\sigma_1 t+\delta_\pm)]. \tag{5.1}$$

The corresponding results for u and v, as given by (3.9), are

$$\{u,v\} = -\tfrac{1}{2}\epsilon_\pm a(\omega\pm\sigma_1)\{\sin,\cos\}[(\theta+\omega t)\pm(\sigma_1 t+\delta_\pm)], \tag{5.2}$$

where the top and bottom signs correspond with those in (5.1). The vertical velocity at the free surface is given by the linearized boundary condition

$$w = \zeta_t+(\omega^2 r/g)\,u \equiv (2r/l)\,u. \tag{5.3}$$

We also have, from the equation of continuity within the liquid,

$$w_z = -r^{-1}[(ru)_r+v_\theta] \equiv 0, \tag{5.4}$$

so that (5.3) gives w everywhere in the liquid (note that $w/u = 2r/l$, the meridian slope of the container).

We remark that each of the two linearly independent solutions represented by (5.1)–(5.3) corresponds to a rigid-body displacement of the liquid having the displacements

$$\{\xi^{(\pm)},\eta^{(\pm)}\} = \tfrac{1}{2}\epsilon_\pm a\{\cos,\mp\sin\}(\sigma_1 t+\delta_\pm) \tag{5.5}$$

in the non-rotating Cartesian co-ordinate system

$$\{x_1, y_1\} = r\{\cos, \sin\}(\theta + \omega t). \tag{5.6}$$

We find that the linear superposition of the two linearly independent solutions satisfies (4.3) but leaves a remainder of $-\epsilon_+ \epsilon_- \sigma_1 h_0 r \sin(2\sigma_1 t + \delta_+ + \delta_-)$ in (4.2). This implies that

$$h = h_0\left(1 - \frac{r^2}{a^2}\right) + \zeta_{11}^{(+)} + \zeta_{11}^{(-)} - \tfrac{1}{2}\epsilon_+ \epsilon_- h_0 \cos(2\sigma_1 t + \delta_+ + \delta_-), \tag{5.7}$$

together with the sum of the velocities given by (5.2), will satisfy the non-linear equations (4.2) and (4.3) exactly. Substituting (5.1) into (5.7), we may transform the result to

$$h = h_0\{1 - [(x_1 - \xi)^2 + (y_1 - \eta)^2 - \overline{(\xi^2 + \eta^2)}]/a^2\}, \tag{5.8}$$

where

$$\xi = \xi^{(+)} + \xi^{(-)}, \quad \eta = \eta^{(+)} + \eta^{(-)}, \tag{5.9}$$

and $\overline{(\xi^2 + \eta^2)}$ denotes the mean value of $\xi^2 + \eta^2$.

We may identify (5.8) as a rigid-body motion of the paraboloidal free surface, in which the centre executes a simple-harmonic, elliptical motion in the non-rotating reference frame with a frequency (σ_1) that is independent of rotation. A case of special interest is rectilinear translation, for which we choose

$$\epsilon_+ = \epsilon_- = \epsilon, \quad \delta_+ = \delta_- = 0 \tag{5.10}$$

to obtain

$$\xi = \epsilon a \cos(\sigma_1 t), \quad \eta = 0. \tag{5.11}$$

We also note that setting $\epsilon_- = 0$ ($\epsilon_+ = 0$) yields a clockwise (counterclockwise), circular translation of the centre of the paraboloid.

It is obvious from their construction by superposition that both u and v are simple harmonic. As for the vertical velocity, we have

$$w = \frac{D}{Dt}\left(h + \frac{r^2}{l}\right) = \frac{Dh}{Dt} + \left(\frac{2r}{l}\right) u \equiv \left(\frac{2r}{l}\right) u, \tag{5.12}$$

in agreement with (5.3). We conclude that the finite-amplitude motion departs from a simple harmonic oscillation only in the displacement of the free-surface. This displacement contains the second harmonic (see (5.7)) if and only if $\epsilon_+ \epsilon_- \neq 0$. The motion is strictly simple harmonic if either ϵ_+ or ϵ_- vanishes, as for the aforementioned circular motions.

REFERENCES

BALL, F. K. 1962 An exact theory of simple finite shallow water oscillations on a rotating earth. Proc. Conference on Fluid Mechanics and Hydraulics, University of Western Australia, Nedlands, 1962, p. 293. Pergamon Press.

BALL, F. K. 1963 Some general theorems concerning the finite motion of a shallow rotating liquid lying on a paraboloid. *J. Fluid Mech.* **17**, 240.

FULTZ, DAVE 1962 *Trans. N.Y. Acad. Sci.* **24**, 421–46.

LAMB, H. 1932 *Hydrodynamics*. Cambridge University Press.

MILES, J. W. 1963 Free-surface oscillations in a slowly rotating liquid. *J. Fluid Mech.* (In the press.)

MURTY, T. S. 1962 Gravity modes in a rotating paraboloidal dish. M.S. Dissertation, University of Chicago.

PLATZMAN, G. W. 1962 Unpublished; cited by Fultz *l.c. ante* (1962).